* Website, bCourses up
* Sign up Piazza
* Lab, Discussion start next week
  ↳ SPICE

EE105
Microelectronic Devices and Circuits

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511 Sutardja Dai Hall (SDH)

Linear Time-Invariant (LTI) System

- **Response of a system**

Input

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

Output

$a\,x_1(t) \rightarrow a\,y_1(t)$

$+)\ b\,x_2(t) \rightarrow b\,y_2(t)$

$a\,x_1(t) + b\,x_2(t) \rightarrow a\,y_1(t) + b\,y_2(t)$

- **The system is linear if**

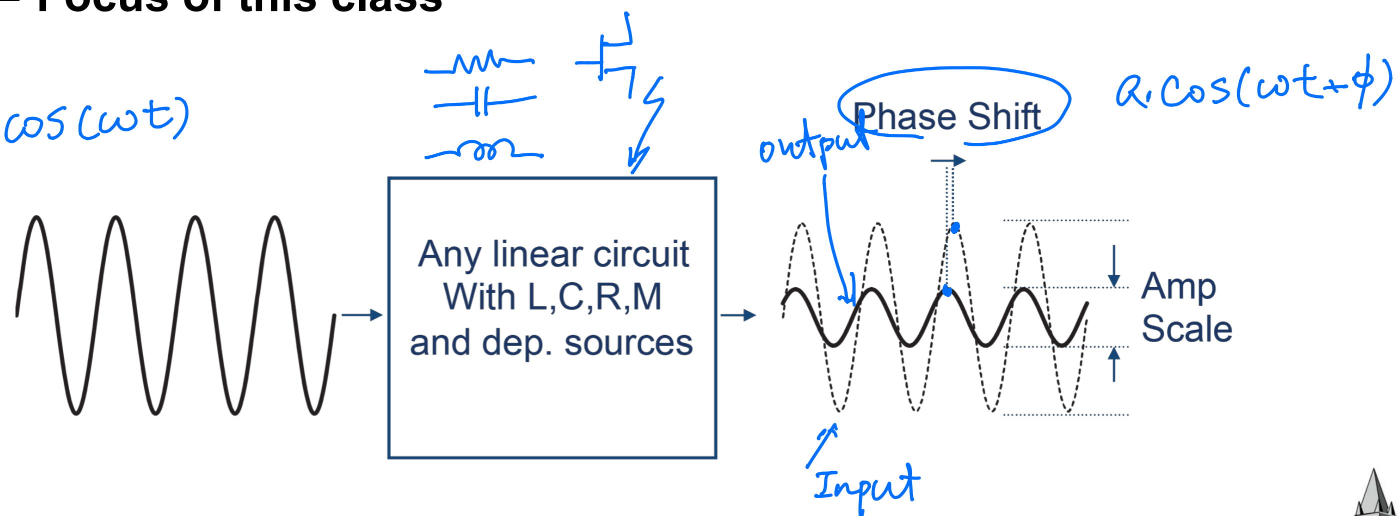
$$a_1x_1(t)+a_2x_2(t) \longrightarrow \boxed{H} \longrightarrow a_1y_1(t)+a_2y_2(t)$$

- **The system is time-invariant if**

$$x(t+T) \longrightarrow \boxed{H} \longrightarrow y(t+T)$$

What's Nice about LTI System?

- **Can use superposition**
- **Easy conversion between time and frequency response**
- **Most systems in real life are LTI systems**
  - **Focus of this class**



Example: Low Pass Filter (LPF)

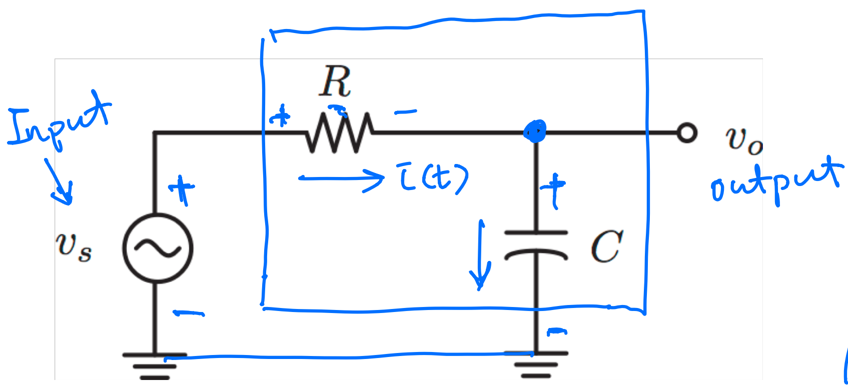
$\omega = 2\pi \cdot f$

$\left[\frac{rad}{s}\right]$ $\quad$ $[Hz] = \left[\frac{1}{s}\right]$

- **Input signal:** $v_s(t) = V_s \cos(\omega t)$
- **We know that:** $v_o(t) = \underbrace{K \cdot V_s}_{V_0} \cos(\omega t + \phi)$

Phase shift

Amp scale



$$v_0(t) = v_s(t) - i(t)R$$

$$i(t) = C\frac{dv_0}{dt}$$

$$v_0(t) = v_s(t) - RC\frac{dv_0}{dt}$$

$$v_s(t) = v_0(t) + \tau\frac{dv_0}{dt} \qquad \tau = RC$$

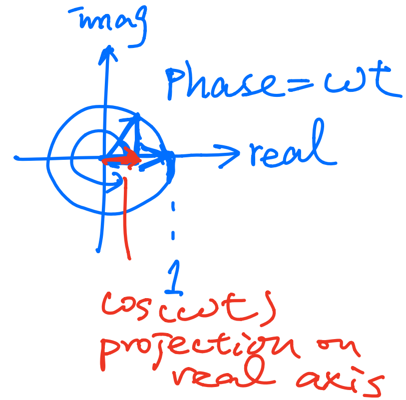
+Q, −Q, C

$Q = C \cdot V$

$[F] = \left[\frac{Coul}{V}\right]$

$i_c = \frac{dQ}{dt} = C \cdot \frac{dV}{dt}$

Exponential Representation



- **Euler's Theorem**

j: imaginary unit

Phase

$$e^{jwt} = cos(wt) + j\,sin(wt)$$

- $\boldsymbol{sin(wt)}$ **and** $\boldsymbol{cos(wt)}$ **can be represented by linear combination of complex exponential:**

$$cos(wt) = \frac{1}{2}\left(e^{jwt} + e^{-jwt}\right)$$

$$sin(wt) = \frac{1}{2j}\left(e^{jwt} - e^{-jwt}\right)$$

Magic: Turn Diff Eq into Algebraic Eq

- Integration and differentiation are trivial with complex numbers:

$$\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t} \qquad \int e^{i\omega\tau}d\tau = \frac{1}{i\omega}e^{i\omega t}$$

- Any ODE is now trivial algebraic manipulations … in fact, we'll show that you don't even need to directly derive the ODE by using phasors
- The key is to observe that the current/voltage relation for any element can be derived for complex exponential excitation

Solving LPF with Phasors

- **Let's excite the system with a complex exp:**

$$v_s(t) = v_0(t) + \tau \frac{dv_0}{dt}$$

$$v_s(t) = V_s e^{j\omega t}$$

$$v_o(t) = |V_0| e^{j(\omega t + \phi)} = V_0 e^{j\omega t}$$

use *j* to avoid confusion

$V_0 = |V_0| e^{j\phi}$

real **complex**

$$V_s \cancel{e^{j\omega t}} = V_0 \cancel{e^{j\omega t}} + \tau \cdot j\omega \cdot V_0 \cancel{e^{j\omega t}}$$

$$V_s = V_0 (1 + j\omega \cdot \tau)$$

Transfer fx

$$\frac{V_0}{V_s} = \frac{1}{(1 + j\omega \cdot \tau)}$$

Easy!!!

Magnitude and Phase Response

- **The system is characterized by the complex function**

Frequency Domain

$$H(\omega) = \frac{V_0}{V_s} = \frac{1}{(1 + j\omega \cdot \tau)}$$

$\tau = RC$

[s] = [Ω][F]

- **The magnitude and phase response match our previous calculation:**

magnitude

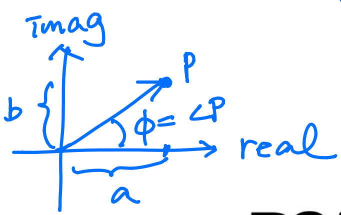
$$|H(\omega)| = \left|\frac{V_0}{V_s}\right| = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

$P = a + jb$

$|P| = \sqrt{a^2 + b^2}$

$\angle P = \tan^{-1}\left(\frac{b}{a}\right)$

$$\prec H(\omega) = -\tan^{-1} \omega\tau$$



Why did it work?

- **Again, the system is linear:**

$$y = \mathbf{L}(x_1 + x_2) = \mathbf{L}(x_1) + \mathbf{L}(x_2)$$

- **To find the response to a sinusoid, we can find the response to $e^{i\omega t}$ and $e^{-i\omega t}$ and sum the results:**

$$e^{i\omega t} \longrightarrow \boxed{\text{LTI System H}} \longrightarrow |H(\omega)|e^{i(\omega t+\phi_1)}$$

$$e^{-i\omega t} \longrightarrow \boxed{\text{LTI System H}} \longrightarrow |H(-\omega)|e^{i(-\omega t+\phi_2)}$$

$$\frac{e^{i\omega t} + e^{-i\omega t}}{2} \longrightarrow \boxed{\text{LTI System H}} \longrightarrow \frac{H(\omega)e^{i\omega t} + H(-\omega)e^{-i\omega t}}{2}$$

cos(ωt)

BSAC

(cont.)

- **Since the input is real, the output has to be real:**

$$y(t)=\frac{H(\omega)e^{i\omega t}+H(-\omega)e^{-i\omega t}}{2}$$

- **That means the second term is the conjugate of the first:**

$H(-\omega)=H(\omega)^*$

$$\left|H(-\omega)\right|=\left|H(\omega)\right| \qquad \text{(even function)}$$

$$\prec H(-\omega)=-\prec H(\omega)=-\phi \qquad \text{(odd function)}$$

- **Therefore the output is:**

$$y(t)=\frac{\left|H(\omega)\right|}{2}\left(e^{i(\omega t+\phi)}+e^{-i(\omega t+\phi)}\right)$$

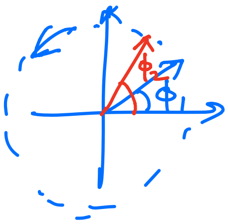
$$=\left|H(\omega)\right|\cos(\omega t+\phi)$$

$x(t)=\cos(\omega t)$

Phasors

$e^{j\omega t+\phi}$ drop $e^{j\omega t}$, track $e^{j\phi}$

- **With our new confidence in complex numbers, we go full steam ahead and work directly with them … we can even drop the time factor $e^{i\omega t}$ since it will cancel out of the equations.**



- **Excite system with a phasor:** $\tilde{V}_1 = V_1 e^{j\phi_1}$
- **Response will also be phasor:** $\tilde{V}_2 = V_2 e^{j\phi_2}$
- **For those with a Linear System background, we're going to work in the frequency domain**
  - **This is the Laplace domain with** $s = j\omega$

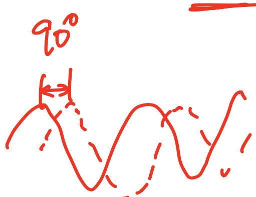
Capacitor I-V Phasor Relation

- **Find the Phasor relation for current and voltage in a cap:**

$nF = 10^{-9} F$

$pF = 10^{-12} F$

$fF = 10^{-15} F$



$$\underline{i_c(t)} = C\frac{dv_C(t)}{dt}$$

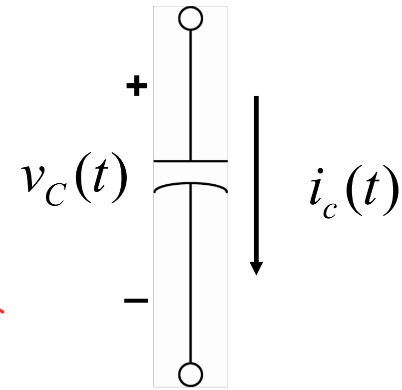
complex

different phase

$$i_c(t) = \underline{I_c e^{j\omega t}}$$

$$v_c(t) = \underline{V_c e^{j\omega t}}$$

complex



$$I_c e^{j\omega t} = C\frac{d}{dt}[V_c e^{j\omega t}]$$

$$CV_c \frac{d}{dt} e^{j\omega t} = \underline{j\omega} CV_c e^{j\omega t}$$

$$I_c \cancel{e^{j\omega t}} = j\omega CV_c \cancel{e^{j\omega t}}$$

$$I_c = j\omega CV_c$$

90° Phase Shift between current & voltage

Impedence

$$\frac{V_c}{I_c} = \frac{1}{j\omega C} = Z$$

Resistor $V = IR$

$$\frac{V}{I} = R$$

$$\frac{1}{[\frac{F}{S}]} = [\frac{S}{F}] = [\Omega]$$

Inductor I-V Phasor Relation

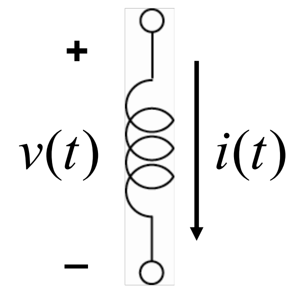
- **Find the Phasor relation for current and voltage in an inductor:**

$$v(t) = L\frac{di(t)}{dt}$$

Inductance [H]

$$i(t) = Ie^{j\omega t}$$

$$v(t) = Ve^{j\omega t}$$



$$Ve^{j\omega t} = L\frac{d}{dt}[Ie^{j\omega t}]$$

$$LI\frac{d}{dt}e^{j\omega t} = j\omega LIe^{j\omega t}$$

$$V\cancel{e^{j\omega t}} = j\omega LI\cancel{e^{j\omega t}}$$

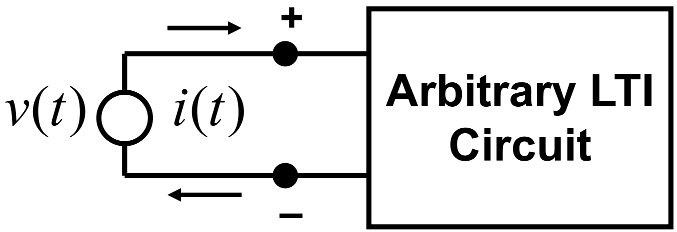
$$V = j\omega LI$$

$$\frac{V}{I} = j\omega L = Z_L$$

$$[\tfrac{1}{s}][H] = [\Omega]$$

Impede the Currents !

- **Suppose that the "input" is defined as the voltage of a terminal pair (port) and the "output" is defined as the current into the port:**



$$v(t) = Ve^{j\omega t} = |V| e^{j(\omega t + \phi_v)}$$

$$i(t) = Ie^{j\omega t} = |I| e^{j(\omega t + \phi_i)}$$

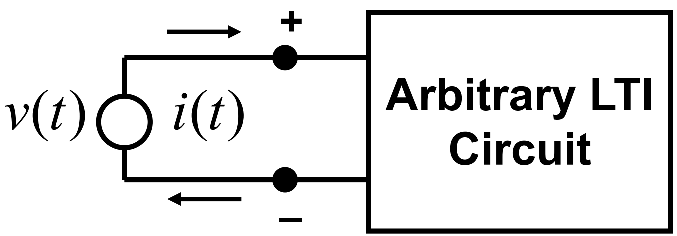
- **The impedance Z is defined as the ratio of the phasor voltage to phasor current ("self" transfer function)**

Complex

$$Z(\omega) = H(\omega) = \frac{V}{I} = \left|\frac{V}{I}\right| e^{j(\phi_v - \phi_i)}$$

Admit the Currents!

- **Suppose that the "input" is defined as the current of a terminal pair (port) and the "output" is defined as the voltage into the port:**



$$v(t) = Ve^{j\omega t} = |V|e^{j(\omega t + \phi_v)}$$

$$i(t) = Ie^{j\omega t} = |I|e^{j(\omega t + \phi_i)}$$

- **The admittance Y is defined as the ratio of the phasor current to phasor voltage ("self" transfer function)**

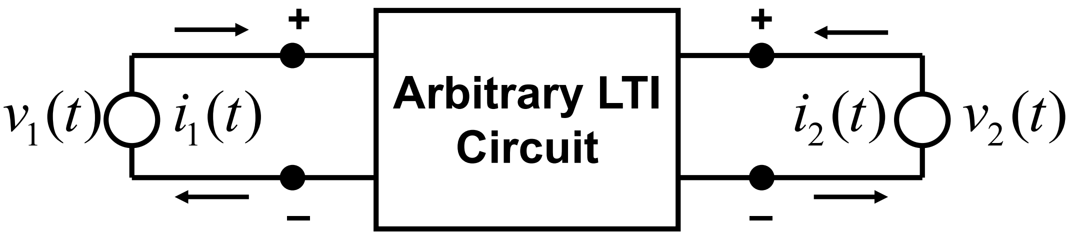
Admittance $$Y(\omega) = H(\omega) = \frac{\dot{I}}{\dot{V}} = \left|\frac{I}{V}\right| e^{j(\phi_i - \phi_v)}$$

[℧] = [S]
mho Siemens

[Ω] Ohm

Voltage and Current Gain

- **The voltage (current) gain is just the voltage (current) transfer function from one port to another port:**



output

$$G_v(\omega) = \frac{V_2}{V_1} = \left|\frac{V_2}{V_1}\right| e^{j(\phi_2 - \phi_1)}$$

input

Voltage Gain: complex

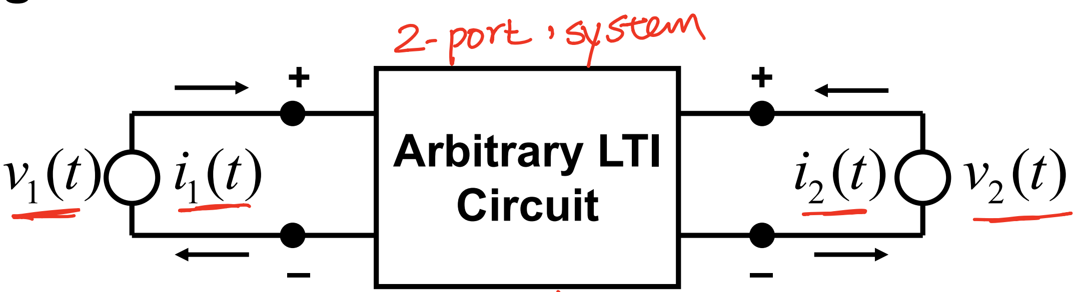
$$G_i(\omega) = \frac{I_2}{I_1} = \left|\frac{I_2}{I_1}\right| e^{j(\phi_2 - \phi_1)}$$

Current Gain: complex

  - **If $|G| > 1$, the circuit has voltage (current) gain**
  - **If $|G| < 1$, the circuit has loss or attenuation**

Transimpedance/admittance

- **Current/voltage gain are unit-less quantities**
- **Sometimes we are interested in the transfer of voltage to current or vice versa**



V: Gain $= \frac{V_2}{V_1}$

I: Gain $= \frac{I_2}{I_1}$

output = voltag

$$J(\omega) = \frac{V_2}{I_1} = \left|\frac{V_2}{I_1}\right| e^{j(\phi_2 - \phi_1)}$$

Input = current

$[\Omega]$ Transimpedence

$$K(\omega) = \frac{I_2}{V_1} = \left|\frac{I_2}{V_1}\right| e^{j(\phi_2 - \phi_1)}$$

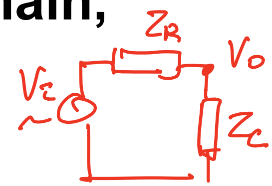
$[S]$ Transconductanc

Direct Calculation of *H* (no DEs)

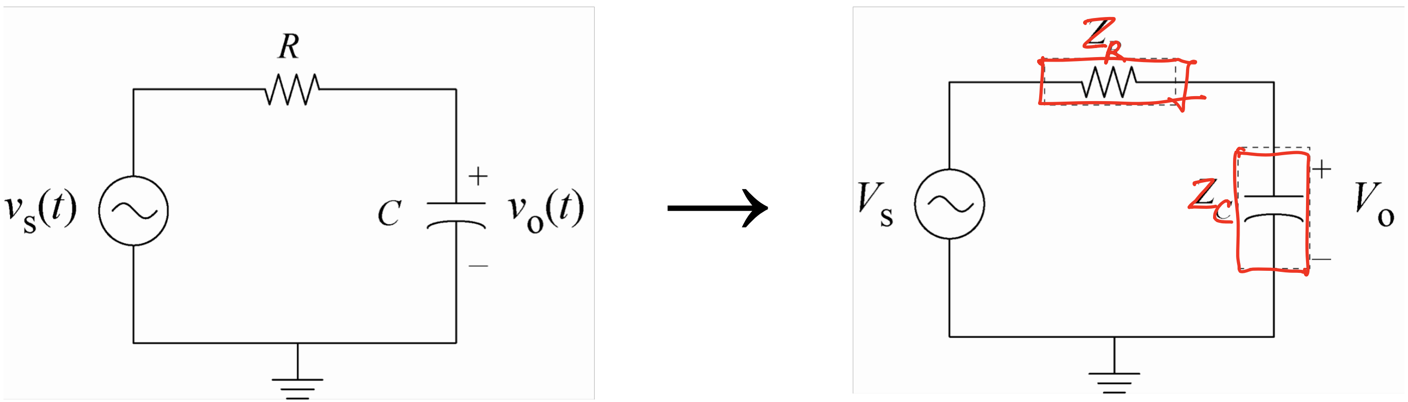
- To directly calculate the transfer function (impedance, trans-impedance, etc) we can generalize the circuit analysis concept from the “real” domain to the “phasor” domain
- With the concept of impedance (admittance), we can now directly analyze a circuit without explicitly writing down any differential equations
- Use KVL, KCL, mesh analysis, loop analysis, or node analysis where inductors and capacitors are treated as complex resistors

LPF Example: Again!

- **Instead of setting up the DE in the time-domain, let's do it directly in the frequency domain**



- **Treat the capacitor as an impedance:**



time domain "real" circuit

frequency domain "phasor" circuit

- **We know the impedances:**

$$Z_R = R \qquad Z_C = \frac{1}{j\omega C}$$

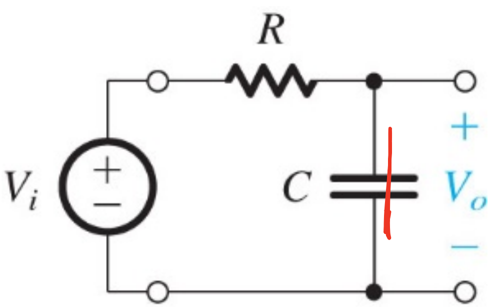
$$V_o = V_i \cdot \frac{Z_c}{Z_R + Z_c}$$

$$H = \frac{V_o}{V_i} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}$$

Bode Plots

- Simply the log-log plot of the magnitude and phase response of a circuit (impedance, transimpedance, gain, …)
- Gives insight into the behavior of a circuit as a function of frequency
- The "log" expands the scale so that breakpoints in the transfer function are clearly delineated

Frequency Response of Low-Pass Filters



$H(\omega)$ $\quad Z_c$

$$T(\omega)=\frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}}=\frac{1}{1+j\omega RC}=\frac{1}{1+j\omega/\omega_0}$$

$RC=\tau$

$$\omega_0=\frac{1}{RC}=\frac{1}{\tau}$$

$$|T(\omega)|=\frac{1}{\sqrt{1+(\omega/\omega_0)^2}}$$

$$\angle T(\omega)=-\tan^{-1}(\omega/\omega_0)$$

$$\omega_{3dB}=\omega_0 \qquad \text{[rad/sec]}$$

$$f_{3dB}=\frac{\omega_0}{2\pi} \qquad \text{[Hz]}$$

Magnitude

$\frac{1}{1+j\frac{\omega}{\omega_0}}$ $\quad \omega\to 0,\ H\to 1$ $\quad (\omega\ll\omega_0)$

$\omega\to\infty\ (\omega\gg\omega_0)$

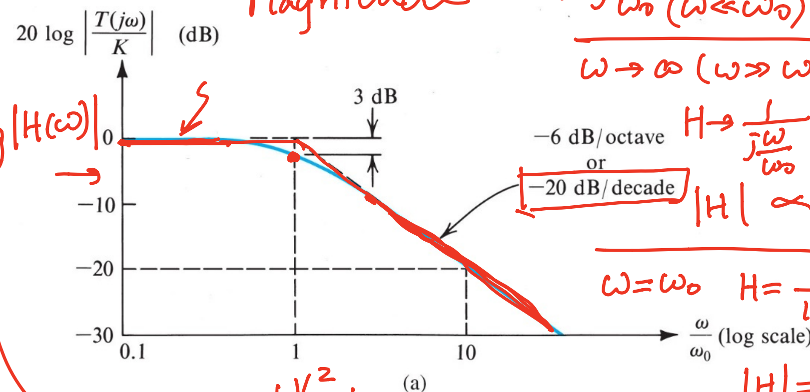
$H\to\frac{1}{j\frac{\omega}{\omega_0}}=\frac{\omega_0}{j\omega}$

$|H|\propto\frac{1}{\omega}$

$\omega=\omega_0 \quad H=\frac{1}{1+j}$

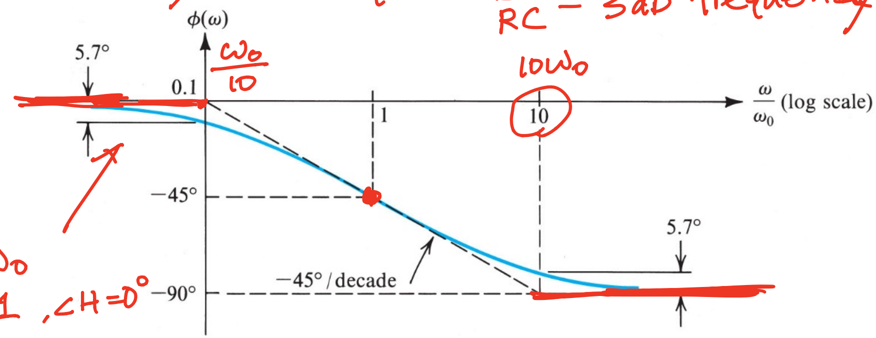
$|H|=\frac{1}{\sqrt{2}}$

$\frac{1}{RC}=$ 3dB frequency



(a)

$10\cdot\log\left|\frac{V_o^2}{V_i^2}\right|$



(b)

$\omega\ll\omega_0$

$H\to 1$, $\angle H=0^\circ$

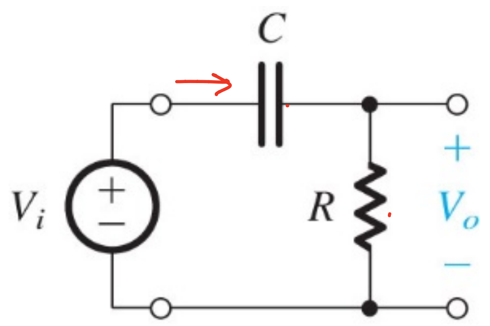
$\omega\gg\omega_0$

$H\to\frac{\omega_0}{j\omega}$ $\quad \angle H=-90^\circ$

$\omega=\omega_0$

$H=\frac{1}{1+j}=1-j$ $\quad \angle H=-45^\circ$

Frequency Response of High-Pass Filters



$$T(\omega)=\frac{R}{R+\frac{1}{j\omega C}}=\frac{1}{1+\frac{1}{j\omega RC}}=\frac{1}{1-j\omega_0/\omega}$$

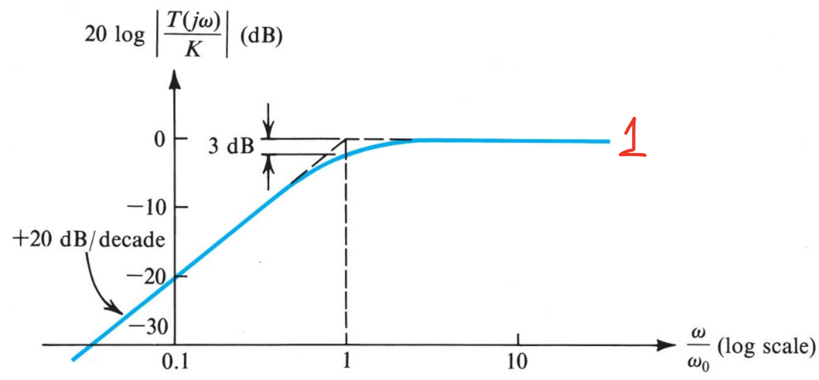
$$\omega_0=\frac{1}{RC}$$

$$|T(\omega)|=\frac{1}{\sqrt{1+(\omega_0/\omega)^2}}$$

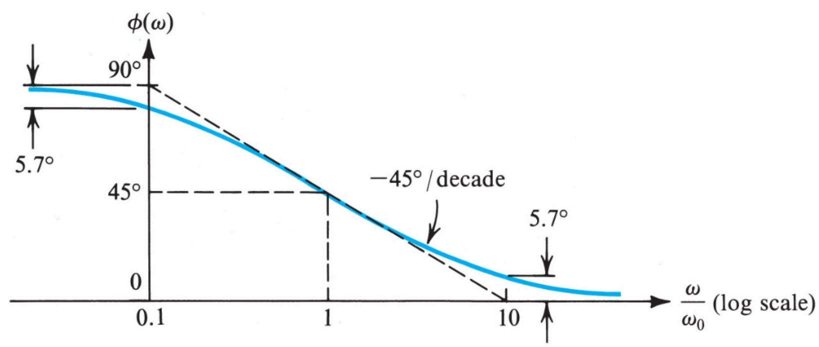
$$\angle T(\omega)=\tan^{-1}(\omega_0/\omega)$$

$$\omega_{3dB}=\omega_0 \qquad \text{[rad/sec]}$$

$$f_{3dB}=\frac{\omega_0}{2\pi} \qquad \text{[Hz]}$$



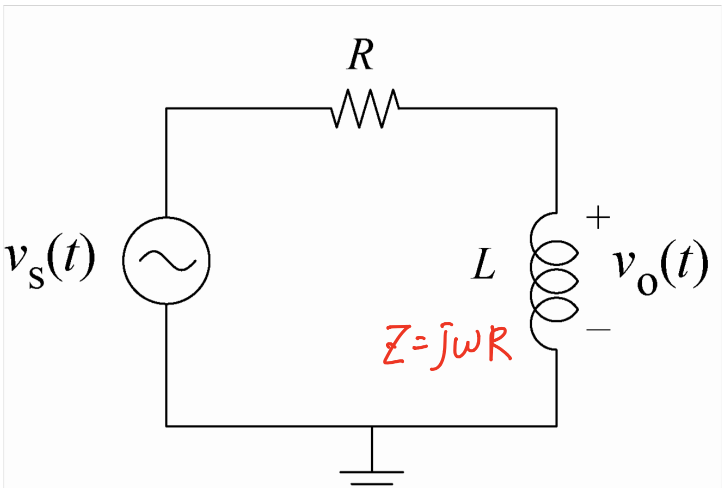
(a)



(b)

Example: High-Pass Filter

- **Using the voltage divider rule:**



$$H(\omega)=\frac{j\omega L}{R+j\omega L}=\frac{j\omega\frac{L}{R}}{1+j\omega\frac{L}{R}}$$

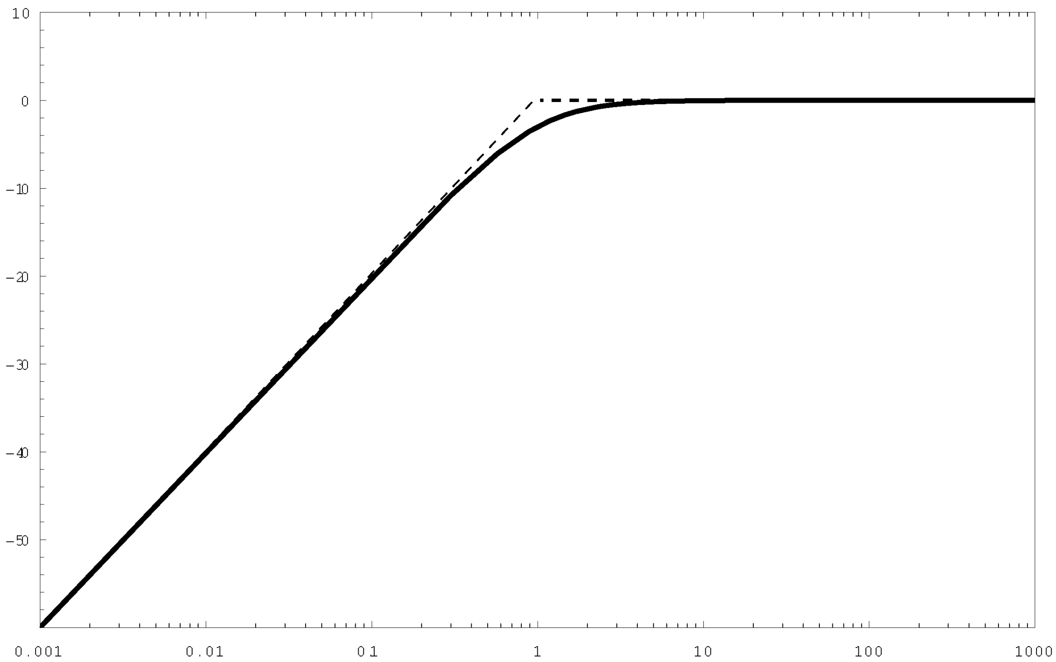
$$H(\omega)=\frac{j\omega\tau}{1+j\omega\tau}$$

$$\omega\to\infty \qquad |H|\to\left|\frac{j\omega\tau}{j\omega\tau}\right|=1$$

$$\omega\to 0 \quad |H|\to\frac{0}{1+0}=0$$

$$\omega=\frac{1}{\tau} \quad |H|=\left|\frac{j}{1+j}\right|=\frac{1}{\sqrt{2}}$$

Approximate versus Actual Plot



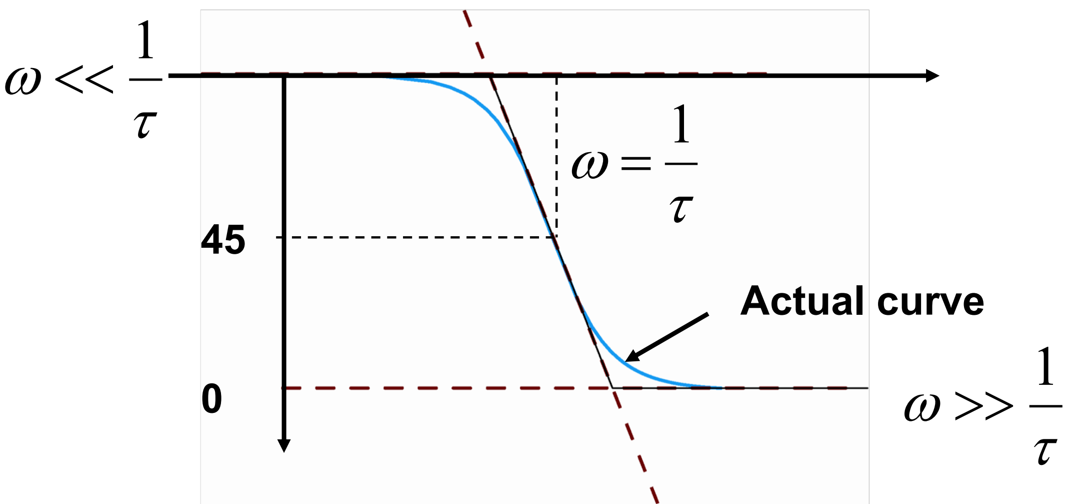
- **Approximate curve accurate away from breakpoint**
- **At breakpoint there is a 3 dB error**

HPF Phase Plot

- **Phase can be naturally decomposed as well:**

$$\prec H(\omega) = \prec \frac{j\omega\tau}{1+j\omega\tau} = \prec j\omega\tau + \prec \frac{1}{1+j\omega\tau} = \frac{\pi}{2} - \tan^{-1}\omega\tau$$

- **First term is simply a constant phase of 90 degrees**
- **The second term is the arctan function**
- **Estimate arctan function:**



Power Flow

- **The instantaneous power flow into any element is the product of the voltage and current:** $P(t) = i(t)v(t)$
- **For a periodic excitation, the average power is:**

$$P_{av} = \int_T \underline{i(\tau)}\,v\underline{(\tau)}\,d\tau$$

$i(t) = |I| \cdot \cos(\omega t + \phi_i)$

$v(t) = |V| \cos(\omega t + \phi_v)$

Power = I × V

- **In terms of sinusoids we have**

$$P_{av} = \int_T |I| \underline{\cos(\omega t + \varphi_i)}|V| \underline{\cos(\omega t + \varphi_v)}\, d\tau$$

$$= |I| \cdot |V| \int_T (\cos\omega t \cos\varphi_i - \sin\omega t \sin\varphi_i) \cdot (\cos\omega t \cos\varphi_v - \sin\omega t \sin\varphi_v) d\tau$$

$$= |I| \cdot |V| \int_T d\tau \cos^2 \omega t \cos\varphi_i \cos\varphi_v + \sin^2 \omega t \sin\varphi_i \sin\varphi_v + c \sin\omega t \cos\omega t$$

$$= \frac{|I| \cdot |V|}{2} (\cos\varphi_i \cos\varphi_v + \sin\varphi_i \sin\varphi_v) = \frac{|I| \cdot |V|}{2} \underbrace{\cos(\varphi_i - \varphi_v)}$$

Power Flow with Phasors

$$P_{av} = \frac{|I| \cdot |V|}{2} \cos(\phi_i - \phi_v)$$

Power Factor

- **Note that if** $(\boldsymbol{\phi_i} - \boldsymbol{\phi_v}) = \frac{\boldsymbol{\pi}}{\mathbf{2}}$ **, then** $\boldsymbol{P_{av}} = \frac{\boldsymbol{|I| \cdot V}}{\mathbf{2}} \boldsymbol{cos}\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right) = \mathbf{0}$
- **Important: Power is a non-linear function so we can't simply take the real part of the product of the phasors:**

$$P \neq \mathrm{Re}[I \cdot V]$$

- **From our previous calculation:**

$$P = \frac{|I| \cdot |V|}{2} \cos(\phi_i - \phi_v) = \frac{1}{2}\mathrm{Re}[I \cdot V^*] = \frac{1}{2}\mathrm{Re}[I^* \cdot V]$$

BSAC

Summary

- **Complex exponentials are eigen-functions of LTI systems**
  - **Steady-state response of LCR circuits are LTI systems**
  - **Phasor analysis allows us to treat all LCR circuits as simple “resistive” circuits by using the concept of impedance (admittance)**
- **Frequency response allows us to completely characterize a system**
  - **Any input can be decomposed into either a continuum or discrete sum of frequency components**
  - **The transfer function is usually plotted in the log-log domain (Bode plot) – magnitude and phase**
  - **Location of poles/zeros is key**

$V = IR$

$Z_C = \frac{1}{j\omega C}$ $\qquad$ $V_C = I_C \cdot Z_C$

$Z_L = j\omega L$ $\qquad$ $V_L = I_L \cdot Z_L$